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Kant's Theory of Geometrical Reasoning and the Analytic–Synthetic Distinction. On Hintikka's Interpretation of Kant's Philosophy of Mathematics

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1. Modern Interpretations of Kant's Philosophy of Mathematics

The thesis that mathematical judgements are not only *a priori* but also synthetic is central to Kant's philosophy of mathematics. To be somewhat more precise: this is Kant's view on the nature of (pure) mathematics in his critical period. Indeed, in the *Prolegomena*—and the same passage returns in the introduction in the second edition of the *Kritik der reinen Vernunft*—one reads:

> All mathematical judgements, without exception, are synthetic. This fact, though incontestably certain and in its consequences very important, has hitherto escaped the notice of those who are engaged in the analysis of human reason, and is, indeed, directly opposed to all their conjectures. For as it was found that all mathematical inferences proceed in accordance with the principle of contradiction (which the nature of all apodeictic certainty requires), it was supposed that the fundamental propositions of the science can themselves be known to be true through that principle. This is an erroneous view. For though a synthetic proposition can indeed be discerned in accordance with the principle of contradiction, this can only be if another synthetic proposition is presupposed, and if it can then be apprehended as following from this other proposition; it can never be so discerned in and by itself (B 14; cf. IV 268).[1]

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[1]The *Kritik der reinen Vernunft* will be referred to in the customary way by the pagination of the first (A) and/or second (B) printing. References to Kant's other works are to the well-known *Akademie Text-Ausgabe* of *Kants Gesammelte Schriften*. Roman numerals indicate the volume and the following Arabic numbers the page(s) in the collected works of Kant. My citations use as far as possible the presently appearing *The Cambridge Edition of the Works of Immanuel Kant*, ed. by P. Guyer and A. W. Wood: *Theoretical Philosophy, 1755–1770*, transl. and ed. by D. Walford and R. Meerbote (Cambridge University Press, 1992); *Lectures on Logic*, transl. and ed. by J. M. Young, (Cambridge University Press, 1993). Other translations used: *Immanuel Kant's Critique of Pure Reason*, transl. by N. Kemp Smith (London: MacMillan, 1978); I. Kant, *Prolegomena to any Future Metaphysics That Will be Able to Present Itself as a Science*, transl. by P. G. Lucas (Manchester University Press, 1971).

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Brittan believes that in this passage Kant is opposing Leibniz in particular, albeit without mentioning his name in so many words.[2] Certainly Leibniz regarded the principle of contradiction or identity as the great foundation of mathematics; or as he presents this conception in his second letter to Clarke: 'this single principle is sufficient to demonstrate every part of arithmetic and geometry, that is, all mathematical principles'.[3] Yet Hume's name too should certainly be mentioned here; for Kant refers to his views explicitly in the *Prolegomena* precisely in this connection (IV 272).

As we saw, in Kant mathematical judgements are not analytic but synthetic (*a priori*). And, quite compatible with that: the principle of contradiction would be 'the common principle of all analytic judgements' (IV 277):

> *The principle of contradiction* must therefore be recognized as being the universal and completely sufficient *principle of all analytic knowledge* (A 151/B 191).

For the distinction between analytic and synthetic propositions, one generally relies today on the so-called Fregean interpretation of analyticity: a proposition is analytic (*a priori*) if and only if its truth can be proved with the help just of general laws of logic and definitions. Frege claims that this characterization satisfies what Kant would have meant by the notion of analyticity.[4] The principle of contradiction (and/or identity) (henceforth PNC) has since ancient times been the most fundamental principle of (general) logic. It is not otherwise in Kant. Thus it is in every respect understandable too that most present-day philosophers support (or, as the case may be, follow) Frege in his interpretation of Kant's notion of analyticity.[5] And next the statement 'all mathematical inferences proceed in accordance with the principle of contradiction' (in the rather long citation with which I began this article) is generally read as saying that inference in mathematics should always be logical inference, i.e. reasoning according to (general) logical rules.[6]

This reading of the passage in question (B 14) is in conflict, however, with the prevalent interpretation of Kant's philosophy of mathematics; this is expressed by Bertrand Russell as follows:

> Kant having observed that the geometers of his day could not prove their theorems by unaided argument, but required an appeal to the figure, invented a theory of

[2]G. G. Brittan, *Kant's Theory of Science* (Princeton, NJ: Princeton University Press, 1978), p. 46.

[3]G. F. Leibniz, *Philosophical Papers and Letters*, ed. and transl. by L. Loemker (Dordrecht: Reidel, 1976), p. 677.

[4]G. Frege, *Die Grundlagen der Arithmetik. Eine logisch-mathematische Untersuchung über den Begriff der Zahl* (Hildesheim: Olms, 1961), pp. 3, 99–100. Here Frege appeals explicitly to the passage in Kant cited at the beginning of this section.

[5]R. C. S. Walker, *Kant* (London: Routledge and Kegan Paul, 1978), pp. 23–24. So also W. v. O. Quine in his renowned and exceptionally influential article, 'Two Dogmas of Empiricism', in *From a Logical Point of View* (New York: Harper & Row, 1961), pp. 20–47.

[6]*Op. cit.*, note 2, pp. 55, 62. L. W. Beck, *Studies in the Philosophy of Kant* (Indianapolis: Bobbs-Merrill, 1965), pp. 89–90.

> mathematical reasoning according to which the inference is never strictly logical, but always requires the support of what is called 'intuition'.[7]

This perspective depends mainly on statements Kant made in the first *Critique* in the section about the transcendental doctrine of method. There Kant asserts, speaking of 'the method of attaining apodeictic certainty which is called *mathematical*' in distinction from that of philosophy or metaphysics:

> *Philosophical* knowledge is the *knowledge gained by reason from concepts*, mathematical knowledge is the knowledge gained by reason from the *construction* of concepts. To *construct* a concept means to exhibit *a priori* the intuition which corresponds to the concept (A 713/B 741; cf. A 817/B 865).

Kant clearly regards the construction of mathematical concepts as something directly related to the exhibition of intuitions (*a priori*). And he proceeds immediately to place this special role of intuitions self-evidently in the framework of the method of proof of mathematics, and in particular of geometry: '... through a chain of inferences guided throughout by intuition, he (i.e. the geometer) arrives at a fully evident and universally valid solution of the problem' (A 717/B 745). Moreover, it is the use of intuitions, i.e. the use of constructions, that makes mathematics in general, and geometry in particular, synthetic (cf. IV 272).

Such an interpretation leads all too easily to a negative and averse assessment of Kant's philosophy of mathematics. This is the result of a complex (and not always equally transparent) interplay of at least two considerations. On the one hand, the idea gradually gained ground—as Russell's statement cited above also indicates—that Kant allowed himself to be misled by the informal and somewhat imprecise way in which Euclidean geometry was presented in his day. Only at the end of the 19th century did the opportunity appear to develop geometry, in part through the pioneering work of David Hilbert, as an adequately rigorously formalized axiomatic theory. On the other hand, and not unconnected with this, the work of Gottlob Frege, in particular, in about the same period made available a precisely formulated formal logic that is powerful enough to do justice to reasoning in mathematics. The logic upon which Kant relied is in the nature of the case still the traditional Aristotelian logic. This logic, in contrast to modern Fregean logic, does not offer the possibility of recognizing that every correct mathematical inference is a form of logical inference. It is simply too weak for that.[8]

[7]B. Russell, *Introduction to Mathematical Philosophy* (London: Allen & Unwin, 1967), p. 145. Cf. M. Friedman, *Kant and the Exact Sciences* (Cambridge, MA: Harvard University Press, 1992), p. 80: 'The present approach to Kant's theory of geometry follows Russell in assuming that construction in pure intuition is primarily intended to explain mathematical proof or reasoning, a type of reasoning which is therefore distinct from logical or analytical reasoning' (*ibid.*, p. 56).

[8]Cf. for example J. M. Young, 'Functions of Thought and the Synthesis of Intuitions', in P. Guyer (ed.), *The Cambridge Companion to Kant* (Cambridge University Press, 1992), pp. 101–122, see p. 106.

In consequence of various considerations, Kant's philosophy of mathematics is often regarded in our century as 'at best out-moded and old-fashioned'.[9] In recent decades, however, it has been repeatedly and correctly argued that this representation of matters constitutes a far too facile dismissal of Kant's philosophy of mathematics:

> (Such a) sketch is not completely misleading. There is some basis for it both in Kant's text and in the history of mathematics. But at the same time, the sketch obscures much of what is interesting in Kant's view.[10]

In his important study *Kant's Theory of Science* Brittan distinguishes two (types of) interpretations of Kant's philosophy of mathematics with are of more interest to a historian of philosophy. These are distinguished as the 'Beth–Hintikka' and the 'Lambert–Parsons reconstruction' (*op. cit.*, pp. 49–66). Both interpretations make use of modern Fregean logic (at least first-order predicate logic) and post-Kantian insights in the field of philosophy of mathematics, and are in that sense just reconstructions. Nevertheless, both reconstructions also claim to afford, at least in principle, a reliable interpretation and clarification of Kant's position. The Beth–Hintikka interpretation is primarily oriented to geometry, while in the Lambert–Parsons interpretation, arithmetic is more prominent.

The focus of this article is Kant's philosophical and methodological views regarding geometry. Pure mathematics as Kant sees it includes not only geometry but also arithmetic (number theory as well as algebra); and he alludes several times too to pure mechanics, or kinematics, as a branch of pure mathematics (II 397, cf. IV 283). What Kant says about arithmetic is more problematical and generally less easy to place and understand than are his views on geometry. The fact that in his day arithmetic, unlike geometry, had not yet been elegantly systematized and formulated as a comprehensive scientific theory seems to be at least partly responsible for this. It can also hardly be surprising, then, that Kant, as is generally agreed, took Euclidean geometry as the paradigm of a mathematical theory.

In this paper I will discuss some aspects of Hintikka's well-known interpretation of Kant's theory of mathematical method and the analytic–synthetic distinction. Section 2 sketches the main lines of Hintikka's interpretation of Kant's philosophy of mathematics. In this interpretation, the mathematical method of proof and the analytic–synthetic distinction are prominent. Their role can only be adequately placed and understood, as I see it, in the light of Kant's view of the structure of a (rational) science. This structure and associated conception of scientific rationality is discussed in Section 3 in a very

[9] *Op. cit.*, note 2, p. 43. Cf. J. Bennett, *Kant's Analytic* (Cambridge University Press, 1966), p. 4.
[10] *Op. cit.*, note 2, p. 44. Cf. J. Hintikka, *Knowledge and the Known. Historical Perspectives in Epistemology* (Dordrecht: Reidel, 1974) (henceforth: *KK*), pp. 129–30, 161. Friedman, *op. cit.*, note 7, p. 55. C. Parsons, 'Kant's Philosophy of Arithmetic', *Mathematics in Philosophy. Selected Essays* (Ithaca: Cornell University Press, 1983), pp. 110–149, see p. 126.

general way. There follows, in Section 4, an investigation of Kant's view of the method of proof of geometry. With that, we have in hand the conceptual framework for evaluating critically the way in which Hintikka connects the distinction analytic–synthetic with the method of proof of mathematics. This evaluation unfolds in Sections 5–7. In these sections I also return to the 'particularly perplexing passage' B 14 (in a given case IV 268) which Hintikka too endeavoured to elucidate.[11] However, his solution of the problem raised by this passage seems less than convincing. My own overall thesis is gathered up in the concluding section, Section 8. Starting from a general characterization of Kant's view of *a priori* science, I argue that the connection between the Kantian conception of mathematical demonstration and the analytic–synthetic distinction, is not what Hintikka suggested it to be, but must be viewed in another way.

2. Hintikka's Interpretation of Kant

Hintikka has developed and presented his view of Kant's philosophy of mathematics and of the Kantian distinction analytic–synthetic in numerous publications. His interpretation of Kant is well known, and he has always acknowledged the seminal role of Evert W. Beth in this regard.[12] Brittan too refers, as we saw, to the 'Beth–Hintikka-reconstruction' and describes Hintikka's interpretation as an 'extension' of Beth's.[13] If I understand him correctly, Hintikka has adopted roughly the following insights and conclusions of Beth's and elaborated them at several points:

(1) The Kantian distinction between analytic judgements and synthetic judgements should be understood in terms of the distinction between analytic and synthetic method(s).[14]

(2) What Kant says concerning the difference between the method of philosophy or metaphysics and that of mathematics in the part of the first *Kritik* that deals with the doctrine of the transcendental method agrees in large measure with what the pre-critical Kant said about this subject, namely, in the *Untersuchung über die Deutlichkeit der Grundsätze der natürlichen Theologie und der Moral* (1764).[15]

[11]Cf. J. Hintikka, *Logic, Language-Games and Information. Kantian Themes in the Philosophy of Logic* (Oxford: The Clarendon Press, 1973) (henceforth: *LLGI*), pp. 214–215. *KK* 173.

[12]*LLGI* 111, 135, 145, 216; *KK* 182n., 198. J. Hintikka, 'Kantian Intuitions', *Inquiry* **15** (1972), 341–345; p. 342.

[13]*Op. cit.*, note 2, p. 53. Cf. Parsons, *op. cit.*, note 10, p. 113, 123–124; Friedman, *op. cit.*, note 7, p. 56n.; *KK 198*.

[14]Cf. *op. cit.*, note 1, p. 55; J. Peijnenburg, 'De Kant-Interpretatie van Evert Willem Beth', *Algemeen Nederlands Tijdschrift voor Wijsbegeerte* **83** (1991), 114–128, see p. 118, 125. For the rest, one comes across this thesis in Beth *expressis verbis* especially in his earlier work. Cf. E. W. Beth, 'Over Kant's onderscheiding van synthetische en analytische oordelen', *De Gids* **106** (1942), 47–62, see p. 55.

[15]*KK* 164, 182n; *LLGI* 145n. Cf. E. W. Beth, *The Foundations of Mathematics. A Study in the Philosophy of Science* (Amsterdam: North-Holland, 1959), p. 41. *Ibid.*, 'Kants Einteilung der Urteile in analytische und synthetische', *Algemeen Nederlands Tijdschrift voor Wijsbegeerte en Psychologie* **46** (1953–54), 253–264, see p. 255.

(3) The method of proof of mathematics is, according to (both the critical and the pre-critical) Kant marked by an appeal to individual objects (or, as the case may be, intuitions, i.e. individual representations).[16]

(4) This method of proof, i.e. this use of individual objects in reasoning, can be reconstructed as logically correct within modern logic (especially: quantification theory or first-order predicate logic).[17]

I shall limit my observations here, however, to the way in which Hintikka seems to have arrived at his interpretation and placement of the distinction between analytic and synthetic judgements in the critical Kant.[18] Thesis 1 plays a central role. As seen by Hintikka, this thesis implies the interpretation of the difference between analytic and synthetic judgements in terms of the different methods according to which these two sorts of truths would be proved. Such an approach introduces a restriction of the distinction analytic–synthetic to the terrain of *a priori* judgements or, to formulate it differently: it situates the distinction analytic–synthetic primarily within the rational sciences, such as mathematics and philosophy. It is precisely this perspective which connects theses 1, 2 and 3 in a very direct way with each other. For in the pre-critical writing *Über die Deutlichkeit* it is stated that the method of mathematics is synthetic and not analytic (II 308; cf. II 276–277). And, according to the critical Kant, (real) mathematical judgements are synthetic judgements (*a priori*). The connection of the distinction between analytic and synthetic judgements (*a priori*) to the distinction between the analytic and the synthetic method is thereupon an intrinsic one insofar as method may simply be construed as a method of proof; i.e. the way in which judgements are demonstrated. Moreover, Hintikka himself generally does not speak of an analytic vs synthetic method, but more specifically of an analytic vs synthetic 'argument steps' (cf. *KK* 135–137, 153–155. *LLGI* 136, 145, 207).

If we restrict ourselves to Kant's philosophy of mathematics—as Hintikka does for the most part—then this representation of matters seems fairly plausible. Yet in the light of Kant's entire philosophical edifice a serious difficulty immediately presents itself; namely, *mutatis mutandis* a number of things seem to fit in much less well for metaphysics or philosophy. For while the pre-critical Kant holds that this discipline would employ the analytic and not

[16] *LLGI* 114–115, 206. Cf. *op. cit.*, note 2, pp. 51, 55. Parsons, *op. cit.*, note 10, p. 113. Peijnenburg, *op. cit.*, note 14, pp. 125–126.

[17] *LLGI* 118–119, 210–211, 216–217. Cf. Parsons, *op. cit.*, note 10, pp. 123–124. Friedman, *op. cit.*, note 10, p. 65n: 'Hintikka argues that Kant's analytic/synthetic distinction is drawn *with*in what we now call quantification theory, and Hintikka calls a quantificational argument *synthetic* when (roughly) new individuals, are introduced. See also *op. cit.*, note 2, pp. 51–53. E. W. Beth, *Aspects of Modern Logic* (Dordrecht: Reidel, 1970), esp. chap. IV. Cf. Peijnenburg, *op. cit.*, note 14, p. 126.

[18] Beth's interpretation and placement of this distinction in Kant is not discussed further here. Peijnenburg argues that Hintikka's interpretation cannot properly be called an extension of Beth's interpretation because Hintikka would not have adopted Beth's view of the Kantian distinction between analytic and synthetic judgements. Cf. Peijnenburg, *op. cit.*, note 14, pp. 126, 128.

the synthetic method, the critical Kant still maintains that 'properly metaphysical judgements are without exception synthetic' (IV 273; cf. B 18, VIII 232–233). And with that the suggestion is engendered that thesis 1 in particular may be built on sand.

Central in Hintikka's interpretation of the Kantian distinction analytic–synthetic is the method by which mathematical judgements, especially propositions within Euclidean geometry, would be proved. And this theme is indeed linked by Kant to the distinction between the mathematical and the philosophical method. In the first *Kritik* Kant contrasts these two with each other as follows:

> Philosophy confines itself to universal concepts; mathematics can achieve nothing by concepts alone but hastens at once to intuition, in which it considers the concept *in concreto*... (A 715/B 743; cf. A 837/B 865, IV 281).

And also:

> ... philosophical knowledge... has always to consider the universal *in abstracto* (by means of concepts), mathematics can consider the universal *in concreto* (in the single intuition)... (A 734–735/B 762–763).

It is just such passages that have led Hintikka (correctly, as I see it, and in the line of Beth) to conclude that Kant in the first *Kritik* reverts to and makes use of insights from the pre-critical writing *Über die Deutlichkeit* (cf. thesis 2). There, for example, Kant already observes that in philosophy 'the universal must rather be considered *in abstracto*' (II 279). Similarly:

> ... mathematics, in its inferences and proofs, regards its universal knowledge under signs *in concreto*, whereas philosophy always regards its universal knowledge *in abstracto*... (II 291; cf. II 278, 292)

The way in which Kant reassumes this distinction in the first *Critique* is essentially based on the opposition between intuition and concept. Speaking about representations (*Vorstellungen*) or cognitions, (the critical) Kant distinguishes between intuitions (*Anschauungen*) and concepts (*Begriffe*) (A 320/B 376–377; cf. XVI 85–88, XX 325). These differ from each other in two respects:

(a) an intuition is a singular, a concept is a general representation (cf. XXIV 616, 904, IX 91);
(b) different from intuitions, concepts represent their object not immediately but mediately (ultimately via intuitions) (A 68/B 93; cf. A 19/B 33).

This second difference concerns immediacy and connects intuitions to sensibility and time and space as general forms of intuition. Hintikka, however, is primarily interested in the first difference.[19] By saying that an intuition is a

[19] *KK* 130; *LLGI* 218–219. Parsons, and subsequently also Thompson, have noted that Hintikka restricts himself to this first aspect of the role of intuitions and that he does not distinguish its second characteristic or else unjustifiably ignores this second aspect. Cf. Parsons, *op. cit.*, note 10, pp. 112–113. M. Thompson, 'Singular Terms and Intuitions in Kant's Epistemology', *Review of Metaphysics* **26** (1972/73), 314–343, see p. 314.

singular representation, Kant means that an intuition always represents one individual object; sometimes he even suggests that an intuition *is* an individual object. It is this appeal to individual objects that would be characteristic for what Hintikka refers to as synthetic argument steps in distinction from analytic argument steps (cf. thesis 3). And it is mainly this form of reasoning *in concreto* that determines the direction of Hintikka's interpretation of Kant's philosophy of mathematics, in any case insofar as the logical point of view is concerned. Before delving into this, I first need to consider Kant's conception of rational science.

3. The Aristotelian Model of Science

In the preface to the 2nd edition of the first *Critique*, Kant in speaking of the possibility of metaphysics as science praises Wolff for providing the example of

> how the secure progress of a science is to be attained only through orderly establishment of principles, clear determination of concepts, insistence upon strictness of proof, and avoidance of venturesome, non-consecutive steps in our inferences (B xxxvi).

With this statement Kant not only demands attention for the systematic characteristics of science but also makes clear that his view of science is embedded in a specific concept of science. This is the model of scientific knowledge as *cognitio ex principiis* (cf. A 836/B 864). In fact it concerns here an ideal of rationality that already appears in Aristotle's notion of apodeictic science. This concept of science is exceptionally influential in traditional philosophy. Wolff, Kant and many others base their definition of science on this concept. As Kant said in his *Metaphysische Anfangsgründe der Naturwissenschaft*: 'Every doctrine, if it is a system, i.e. a body of knowledge organized according to principles, is called a science'.[20] Kant considers it to be more or less self-evident that every *a priori* science, not only geometry but also arithmetic, pure natural science and even metaphysics has to meet, at least ideally, the requirements of this concept of scientific rationality. Henceforth I shall refer to this concept as the Aristotelian model of science. A scientific theory conforming to this model will be referred to as an Aristotelian science. Euclidean geometry has long stood as the most perfect realization of an Aristotelian science.

Scholz and in his steps Beth has attempted to define this model of science in a precise way. Here is an adapted characterization of such a science:[21]

[20]IV 467; cf. p. 468: 'One can speak of real science only when there is apodeictic certainty'.

[21]Cf. H. Scholz, 'Die Axiomatik der Alten', in *Mathesis Universalis. Abhandlungen zur Philosophie als strenger Wissenschaft*, ed. by H. Hermes *et al.* (Basel: Schwabe, 1961), pp. 27–44, see pp. 29–30. Beth, *The Foundations of Mathematics*, note 15, pp. 31–32. W. R. de Jong, 'How is Metaphysics as a Science Possible? Kant on the Distinction between Philosophical and Mathematical Method', *The Review of Metaphysics* **48** (1995), 235–274.

An Aristotelian science is a system S of judgements (propositions, statements) and concepts (terms) which satisfies the following conditions:

(1) Each judgement and each concept of S refers to a specific set of objects or domain of reality.
(2) (a) There are in S a (finite) number of so-called fundamental concepts of S.
(b) Any other concept occurring in S may be defined in terms of these fundamental concepts.
(3) (a) S contains a finite number of judgements which are called the fundamental judgements (fundamental propositions) of S.
(b) All other judgements of S may be proved starting from these fundamental judgements.
(4) Any judgement of S must be necessary.

The fundamental judgements (3a) and concepts (2a) of an Aristotelian science are commonly referred to as principles (cf. IV 265). Examples are the concepts of point or line, which are fundamental concepts of Euclidean geometry. Its fundamental judgements are what used to be called its axioms. An example is the proposition that between two points only one straight line is possible (cf. A 163/B 204).

In the case of mathematics, (pure) natural science and metaphysics the judgements or propositions spoken of in condition 3 are qualified by Kant as synthetic judgements *a priori*. Omitting for a moment the case of logic—as Kant himself almost always did when speaking about the *a priori* sciences—one can state that synthetic judgements *a priori* are at the core of the Kantian concept of pure (rational or *a priori*) science, or of the theoretical sciences of reason, as Kant sometimes called them (B 14–19; IV 275–280).

Condition 3b will be referred to as the proof postulate. This postulate is concerned with the systematic coherence and logical ordering of a science on the level of judgements or propositions. It has to guarantee strictness of proof; in this way 'venturesome, non-consecutive steps in our inferences' are to be avoided. Condition 2b, the requirement that concepts ultimately be definable in terms of a limited number of fundamental concepts, satisfies the so-called composition postulate. This postulate guarantees the coherence of a science at the level of concepts or terms. The requirement of 'a clear (*deutliche*) determination of concepts' aims precisely at this postulate. Kant's analytic judgements have their place here too. An analytic judgement satisfies the requirement for a clarification of a concept; it is merely explicative (IV 266). 'By means of... analytic judgements we try to arrive at the definition of a concept' (IV 273). Insofar as the *a priori* sciences are concerned, Kant holds that such a clarification is especially needed for the concepts of metaphysics.

The total coherence of an Aristotelian science is guaranteed in the last analysis by the connection of its concepts and judgements to a specific domain of objects. In the case of geometry this is according to Kant the domain of objects in space. Geometry is nothing else than the mathematics of space (*Mathematik der Ausdehnung*; cf. A 163/B 204); its field is the domain of extended objects (in general). Condition 1 determines the reality postulate, the 'ontological' counterpart of the propositional and conceptual coherence of an Aristotelian science.

Concerning the fourth condition, the necessity postulate, Scholz rightly speaks of 'the most obscure part' of this concept of science.[22] It somehow discounts that scientific knowledge should be apodeictic knowledge. This condition has both epistemic and ontological implications. To be more precise, the followers of the model of science here at issue invariably neglect to distinguish clearly between ontology and epistemology; in any case they do not play them off against each other in the way in which this is often done in present-day philosophy. I will not go into this any further here. At this moment it suffices to note that this postulate implies in any case that scientific knowledge is not only true, but also that it is certain (epistemically necessary). Kant's requirement that scientific knowledge should be *a priori* does justice to the necessity postulate: pure rational science consists of judgements *a priori*; i.e. knowledge which is necessary or apodeictically certain.

Euclidean geometry is normally not only held to satisfy the postulates of an Aristotelian science to a high degree; over and above that, the principles of this discipline are generally deemed to be self-evident, i.e. 'immediately certain' (A 732–733/B 760–761). Following a suggestion of Frege's, I shall refer to this condition as the Euclidean postulate:

An Aristotelian science S satisfies the Euclidean postulate iff,

(a) the fundamental propositions of S are self-evident; i.e. are immediately known to be true and require no further proof.

(b) the fundamental concepts of S are self-evident, i.e. are so clear and well-known as to require no further explanation.

Aristotle already distinguished between sciences that satisfy and sciences that do not satisfy this (epistemological) postulate. The principles of the former he describes not only as better known and prior to nature but also as better known and prior to us. There would of course also be sciences the principles of which are better known and prior to nature but are not better known and prior to us. The qualification 'better known and prior to nature' concerns especially the *ordo essendi*, while 'better known and prior to us' concerns the *ordo cognoscendi*. Physics according to Aristotle does not satisfy the Euclidean postulate, because its principles would still have to be discovered and justified

[22]Cf. Scholz, *op. cit.*, note 21, p. 30.

or proved via sense experience. Such principles are consequently not 'better known and prior to us'.[23] In his own turn Kant makes clear, in speaking of the difference between geometry and philosophy or metaphysics, that geometry does, but that philosophy or metaphysics does not, satisfy the Euclidean postulate. He restricts the use of the term 'axiom' to fundamental judgements which should be intuitively, i.e. immediately, certain. In other words: only fundamental (synthetic) judgements which satisfy the Euclidean postulate—Kant speaks of 'intuitive principles'—are recognized by him as axioms. Fundamental judgements which do not satisfy this postulate he refers to as 'discursive principles' (cf. A 732–733/B 760–761; IX 110).

There is, however, still a further complication. Namely, the venerable term 'principle' has traditionally been used to indicate more than just the fundamental judgements (and fundamental concepts) of an Aristotelian science. This term generally has a broader scope. Aristotle already distinguishes between principles that are proper to a particular science and principles that are common to more than one science. This is a traditional distinction within Euclidean geometry too.[24]

Proper principles obtain only for objects from the domain of a particular science; such principles are specific. The fundamental propositions and concepts of a science are classified as its proper principles. Yet the propositions which characterize or define the other concepts of a science (ideally ultimately in terms of fundamental concepts) are often also referred to as proper principles.[25]

In contrast to a proper principle, a common principle obtains for more than one science. Aristotle identifies as such the PNC, the principle of the excluded middle, and the principle that if equals are subtracted from equals the remainders are equal. These principles have a place in arithmetic as well as in geometry. As a consequence, these principles are common or not domain-specific. Moreover, Aristotle calls these (regulative) principles 'principles of demonstration'; as a result, they are very closely connected with the proof postulate of an Aristotelian science.[26]

In the pre-critical Kant the distinction between common and proper principles appears as that between formal en material principles (of human reason). The principle of identity and the PNC are formal principles. They constitute even 'the supreme universal principles, in the formal sense of the term, of

[23]Cf. Aristotle, *Analytica Posteriora* 71b 33–72a5. R. D. McKirahan, *Principles and Proofs. Aristotle's Theory of Demonstrative Science* (Princeton University Press, 1992), pp. 29–33. In the philosophy of Modern Times this distinction is explicitly advocated again by among others Hobbes in his *Computation or Logic: The English Works of Thomas Hobbes*, ed. by W. Molesworth (Aalen: Scientia 1962), Vol. I, p. 67.

[24]Th. L. Heath (ed.), *The Thirteen Books of Euclid's Elements* (New York: Dover, 1966), Vol. 1, pp. 117–124.

[25]Cf. McKirahan, *op. cit.*, note 23, p. 37; see also A 300/B 356–357.

[26]Cf. Aristotle, *Analytica Posteriora* 76a 36ff. McKirahan, *op. cit.*, note 23, pp. 68–79.

human reason in its entirety' (II 294–295). The critical Kant brings these (regulative) principles into connection with what he calls general logic; i.e. the logic which treats of the 'form of thought in general'; this logic has the PNC as its 'highest principle' (A 54–55/B 78–79; cf. A 131/B 170, A 150/B 189). Broadly speaking, the principles of Kant's general logic function as common principles within, for example, mathematics, (pure) physics and metaphysics. And it is precisely such general logical principles that Kant has in mind when he observes regarding geometry:

> Some few fundamental propositions (*Grundsätze*), presupposed by the geometrician, are indeed, really analytic, and rest on the principle of contradiction. But as identical propositions, they serve only as links in the chain of method and not as principles (i.e. as proper principles or as fundamental judgements—WRdJ); for instance, $a=a$; the whole is equal to itself; or $(a+b)>a$, that is, the whole is greater than its part (B 16–17).

This passage brings me back again to the question we encountered in the first section: the question, namely, of the extent to which Kant would claim, or not claim, that mathematical reasoning coincides with 'strictly logical' reasoning. In the present context, that of the Aristotelian model of science, this question comes down to the following: does the proof-postulate of a mathematical science justify, according to Kant, (types of) inferences which are not of a general logical nature? That it is not entirely inconceivable this question should be answered in the affirmative is already obvious when one recalls that Kant recognizes, besides general logic, the additional possibility of special logics:

> Logic,..., can be treated in a twofold manner, either as logic of the general or as logic of the special employment of understanding. The former contains the absolutely necessary rules of thought without which there can be no employment whatsoever of the understanding. It therefore treats of the understanding without any regard of the difference in the objects to which the understanding may be directed. The logic of the special employment of the understanding contains the rules of correct thinking as regards a certain kind of objects (A 52/B 76).

Insofar as the domain of philosophy or metaphysics is concerned, 'transcendental logic' is introduced by Kant as such a special logic. Our question now is whether there should be a special logic of mathematics (or of geometry). And if so, what is the character of the special rule(s) of inference regarding mathematical (or geometrical) objects.

4. Proofs in Euclidean Geometry

The mathematical method of proof is marked, according to Kant, by reasoning '*in concreto*'. The critical Kant speaks of this as 'the intuitive employment (of reason) by means of the construction of concepts' (A 719/B 747). In *Über die Deutlichkeit* Kant amplifies this procedure of proof as follows:

> ... in geometry, in order, for example, to discover the properties of all circles, one circle is drawn; and in this one circle, instead of drawing all the possible lines which

> could intersect each other within it, two lines only are drawn. The relations which hold between these two lines are proved; and the universal rule, which holds between intersecting lines in all circles whatever, is considered in these two lines *in concreto* (II 278).

It is relatively simple to identify the geometrical insight at issue in this train of thought. Kant is clearly alluding to the proof of proposition 35 from book III of the *Elementa* of Euclid: 'If in a circle two straight lines intersect one another, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other'.[27] This proposition, saying that $AE \times EC = BE \times ED$ (where AE is the length of the segment AE, etc.), is indeed proved through the use of a concrete geometrical figure, in this case a figure (Fig. 1) containing a circle and two lines that intersect one another (I somewhat simplify the actual state of affairs in Euclid; in fact, the proof is one by cases and there are also auxiliary constructions required).

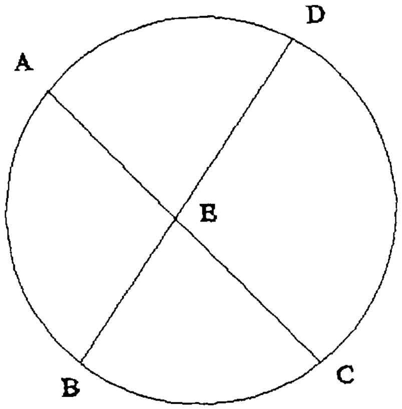

Fig. 1

In the first *Kritik* Kant elucidates this procedure of proof using the renowned proposition 32 of book I of the *Elementa* as a model: in any triangle, if one of the sides be produced, the exterior angle is equal to the two interior and opposite angles, and the three interior angles of the triangle are equal to two right angles.[28] With these examples Kant describes the method of proof of (Euclidean) geometry in a striking way indeed.

According to Proclus the solution of a problem or the proof of a theorem in Euclidean geometry consists ideally of six parts. Some of these always appear,

[27]Cf. *op. cit.*, note 24, Vol. 2, p. 71. Euclid presents this proposition in a strongly geometrically tinted formulation; for him numbers *are* more or less line lengths and the product of two numbers is conceived by him as the surface of a rectangle with those lines as it sides.

[28]*Op. cit.*, note 24, Vol. 1, pp. 316–317. Cf. A 716–717/B 744–745. See also Friedman, *op. cit.*, note 10, pp. 56–58; Beth, *op. cit.*, note 17, p. 42.

others sometimes not. Always appearing are enunciation (πρότασις), proof (ἀπόδειξις) and conclusion (συμπέρασμα). Generally we also find in addition a setting-out or ecthesis (ἔκθεσις), a definition or specification (διορισμός) and a (auxiliarly) construction or machinery (κατασκευή). This description of the procedure of geometry was by no means unknown in the 17th and 18th centuries.[29] Most of these elements also turn up in Kant's description of the method of geometry.

To start with there is the enunciation of a general proposition which has to be demonstrated. Next Euclid usually introduces a concrete geometrical figure; thus in order to prove that a circle, or every circle, as the case may be, features a certain characteristic, an appeal is generally made to an (arbitrary) individual circle, which is almost always drawn concretely. This is called the setting-out or ecthesis. Note that the critical Kant speaks of this step as 'constructing' a geometrical figure. The definition or specification aims to determine more precisely the figure set out in the ecthesis. Moreover, it often proves necessary to expand this setting-out through the use of diverse auxiliary constructions. This is called the construction or machinery. Focusing on proposition 32 of book I of the *Elementa*, Kant describes this part of the procedure as follows: 'he (the geometer) prolongs one side of the triangle and obtains two adjacent angles.... He then divides the external angle by drawing a line parallel to the opposite side of the triangle,...'. (A 716–717/B 744–745). The construction is followed by the so-called apodeixis in which it is proved that the property in question does indeed belong to the geometrical figure introduced in the ecthesis. Finally, as Hintikka puts it, 'having reached the desired conclusion about the particular figure, Euclid returned to the general enunciation again, saying e.g. "Therefore, in *any* triangle, etc." (*KK* 167–169). This part constitutes what Proclus named the conclusion.

Logically considered, the conclusion (συμπέρασμα) is the result of the use of an inference rule which in modern predicate logic (according to the system of natural deduction) is known as the rule of universal generalization (introduction of the universal quantifier): $\alpha(a) \therefore \forall x \alpha(x)$, provided that the individual constant a is arbitrary.This rule indicates when a proof for an individual case may be taken as a basis for deciding that the general case is proven (e.g. a proof holding for any triangle, or a proof holding for any two intersecting lines within any circle). Such a generalization is only permitted if the individual constant concerned is an arbitrary one. This means that in the ecthesis, the setting-out of a particular figure should in principle already take into account this boundary condition; intuitively, this condition for Kant's model proposition from the first *Kritik* comes down to the requirement that

[29]Cf. *op. cit.*, note 24, Vol. 1, pp. 129–131. Commandino's very influential Latin translation of Pappus' *Collectio* appeared in 1589. Cf. G. W. Leibniz, *Nouveaux Essais sur l'Entendement Humain*, Book IV, chap. XVII, §3.

instead of the concretely constructed triangle, any other triangle might have been employed.

In the machinery (κατασκευή), mainly auxiliary constructions are introduced. An appeal is thereby made to inference patterns which Hintikka relates to what is nowadays called the rule of existential instantiation (elimination of the existential quantifier): $\exists x\ \alpha(x),\ \alpha(a)\supset\beta \therefore \beta$, provided that the individual constant a is arbitrary. While in his interpretation of Kant Hintikka associates the characteristic method of Euclidean geometry particularly with the rule of existential instantiation, Beth refers in the first place to the rule of universal generalization.[30] From the standpoint of formal logic, however, these two rules of deduction are closely allied. Moreover, both rules are generally accepted rules of modern standard (predicate) logic, while both fall outside the scope of what is conceived as the standard logic of Kant's day, i.e. syllogistics (including the PNC). In other words, these inference patterns which function prominently within Euclidean geometry do indeed lie outside Kant's so-called general logic. They are patterns which have to be conceived as (regulative) principles belonging to what constitutes according to Kant a 'logic of the special employment of the understanding'.

Remarkably, in speaking of Kant's philosophy of mathematics, Hintikka devotes no attention at all to what Proclus names the definition or specification (διορισμός) (cf. *KK* 168; cf. Section 6 below). Nevertheless, it is natural to assume that the critical Kant would connect not only the machinery and the ecthesis but especially also the definition or specification with constructions and the constructive activities of the geometer. Kant's intuitive–constructivist view of mathematics clearly concerns in the first place these three parts or elements of a demonstration in Euclidean geometry.

5. Analysis vs Synthesis in Kant According to Hintikka

Speaking of the historical background against which Kant supposedly introduced his distinction between analytic and synthetic judgements, Hintikka juxtaposes on the one hand the directional sense, and, on the other, the constructional sense of analysis and synthesis. The opposition analytic–synthetic would go back to ancient Greek mathematics, where it would be found in a particularly difficult to interpret text of Pappus in which he endeavors to elucidate the meaning of 'analysis' and 'synthesis' (as methods in geometry) (cf. *LLGI* 199–205. *KK* 170–171). However, I will restrict myself here as much as possible to the way in which Hintikka uses this distinction in his interpretation of Kant's philosophy of mathematics.

[30] *KK* 154, 175; *LLGI* 114–119. Beth, 'Kants Einteilung der Urteile in analytische und synthetische', note 15, p. 251. Cf. *op. cit.*, note 2, pp. 52–54. Friedman (*op. cit.*, note 10, p. 65) is, however, rather critical of Hintikka's interpretation of the role of Kantian intuitions in mathematical reasoning in terms of the rule of existential instantiation. As I believe this question to be of subordinate importance at the moment, I shall not deal with it any further here.

The directional sense of analysis and synthesis, which already gained some prominence in Aristotle, is characterized by Hintikka as follows:

> On this interpretation, one is proceeding analytically when one follows relations of logical consequence (or, in some cases, relations of causal consequence) in a direction opposite to their normal direction, whereas one is proceeding synthetically when one follows them in their natural direction (*LLGI* 201).

Hintikka signalizes correctly that this interpretation, which he nevertheless construes far too restrictively,

> remained the predominant one, or one of the predominant ones, in philosophical literature all through the Middle Ages.... It applies much more generally than merely in geometry. Its connection with geometry was frequently lost altogether in the history of philosophy (*LLGI* 201).

The directional distinction between analysis and synthesis is from the outset intrinsically connected to what I have called above the Aristotelian model of science. Pappus already explicitly posits that (a form of) analysis evinces a direction toward first principles:

> ... in analysis we assume that which is sought as if it were (already) done (*γεγοναος*), and we inquire what it is from which this results, and again what is the antecedent cause of the latter, and so on, until by so retracing our steps we come upon something already known or belonging to the class of first principles, and such a method we call analysis as being solution backwards (*ἀνάπαλιν λύσιν*).[31]

Analysis waits here as a method (way or route) in the direction of the principles of a science; synthesis has just the opposite direction: from principles towards what is grounded on them. It is this contextualization of the distinction between analysis and synthesis that as I see it is also still very much alive in Kant:

> *Analytic* is opposed to *synthetic* method. The former begins with the conditioned and grounded and proceeds to principles (*a principiatis ad principia*), while the latter goes from principles to consequences or from the simple to the composite (IX 149; cf. II 308, XXIV 291, 481, 598, 779).

It is that final rider in particular that makes it clear, too, that the directional distinction between analysis and synthesis is not just connected to the demonstrative ordering of propositions in an Aristotelian science (the proof postulate)—as Hintikka constantly assumes—, but equally to the conceptual ordering (the composition postulate); i.e., to the composition (definition) or resolution of concepts in terms of ultimately fundamental concepts (recall that fundamental concepts are also, as I have noted, commonly referred to as principles). Not infrequently—and Kant too, I believe, is a clear example of it—this connection to concepts even appears as the more prominent one. This means, then, that when in the context of the distinction between analytic and

[31] *Op. cit.*, note 24, Vol. 1, p. 138.

synthetic method there is talk of the relation of cause and effect or of ground and grounded, these may not generally simply be identified with or interpreted exclusively in terms of a relation of logical inference (consequence).

The distinction between analysis and synthesis in its broad directional sense ought to be understood in the light of the Aristotelian model of science; i.e. of science as a *cognitio ex principiis*. Synthesis in that case waits as a movement of thought from the more universal toward the more individual, from the simple toward the more complex, from causes to consequences. The synthetic method stands for the route of the inference of propositions from principles and the composition of inferred or subordinate concepts from fundamental concepts. Analysis features precisely the opposite direction: that from the individual to the more general, from the whole to the parts, and from consequences or effects to causes.

But I turn to what, in contrast with the directional sense, Hintikka refers to as the constructional sense of analysis and synthesis. Proceeding again from the statements of Pappus, he characterizes this approach as follows:

> On this interpretation, a method or procedure is analytic if in it we do not introduce any new geometrical entities, in brief, if we do not carry out any constructions. A procedure is synthetic if such constructions are made use of, i.e. if new geometrical entities are introduced into the argument (*LLGI*, p. 203; cf. *KK* 171).

In an analytic procedure the matter is primarily one of the interrelations of the different parts of a geometrical figure. Constructing geometrical figures in intuition comes down logically to the introduction of individual objects, and thus it is obvious that the corresponding inferences are called synthetic in this constructional sense. Then, making a leap to the beginning of Modern Times, Hintikka asserts that

> (t)his meaning of the term 'analysis' was naturally extended from the analysis of geometrical configurations to the 'analysis' of physical and astronomical configurations. This is roughly the sense in which the first great modern scientists speak of analysis (*LLGI* 204).

These last remarks are made with Galileo and Newton in mind, among others; although what Newton has to say about the application of the analytic and the synthetic method in physics, for example, fits in rather nicely as I see it with the picture sketched in above of the (broad) directional interpretation of analysis and synthesis embedded in the Aristotelian model of science.[32] I shall leave this question here for what it is and restrict myself to Hintikka's interpretation of the Kantian notion of analyticity in which the constructional sense of the distinction between analysis and synthesis occupies a prominent place. For it is his claim that the following definition of analyticity 'approximates rather closely Kant's notion of analyticity' (*LLGI* 137):

[32]Cf. *LLGI* 204. Cf. I. Newton, *Opticks or a Treatise of the Reflections, Refractions, Inflections and Colours of Light* (New York: Dover, 1979), pp. 404–405.

> An argument step is analytic if and only if it does not introduce any new individuals into the discussion (*LLGI* 136; cf. 175, 206–208).

This implies that an argument step is synthetic if and only if it introduces at least one new individual into the discussion.

Moreover, sometimes Hintikka argues for a more restricted claim concerning Kant's notion of analyticity. He then states that the foregoing definition 'may be taken to be a very good reconstruction of Kant's notion of analyticity as he used it in his philosophy of mathematics' (*LLGI* 182, cf. 145, 207). For the continuation of this article it is not necessary to distinguish between these two claims.

It is on the basis of the structure of proofs in Euclidean geometry (cf. Section 4), as discussed above that, according to Hintikka, the passage which was quoted in Section 1 can be understood. It concerned mainly the following words of Kant:

> ... all mathematical inferences proceed in accordance with the principle of contradiction (which the nature of all apodeictic certainty requires) (B 14).

The problem arose through the connection of this statement with, on the one hand, the view that the PNC should be 'the universal and completely sufficient principle of all analytic knowledge' (cf. A 151/B 191) and, on the other hand, Kant's conviction that mathematical judgements are synthetic *a priori* and that proofs in mathematics, especially in Euclidean geometry, usually make use of what Hintikka calls synthetic argument steps. Hintikka's way out is, however, rather simple:

> The passage becomes very natural if we take Kant for his word and understand him as referring solely to the apodeictic or 'proof proper' part of a Euclidean proposition. Taken literally, the proof proper or apodeixis is after all the only part of a Euclidean proposition where inferences are drawn. Taken in this way Kant's statement expresses precisely what he would be expected to hold on my interpretation, viz. that the distinction between on the one hand *apodeixis* and on the other hand *ecthesis* and the auxiliary construction separates the analytic and the synthetic parts of a mathematical argument (*KK* 173; cf. *LLGI* 206, 214–215).

Although Kant at this place—the introduction to the 2nd edition of the first *Kritik*—in no way alludes to proofs in Euclidean geometry, and consequently also provides no direct occasion for advancing this interpretation, it should be recognized that it is not in conflict with Kant's description of the method of the geometer as a 'reasoning in concreto' (cf. Section 4). But is the way the analytic–synthetic distinction is introduced here convincing? I do not think so, but in order to make this clear, I need first to turn for a moment to the way in which the critical and the pre-critical Kant employs the distinction analysis–synthesis, or analytic–synthetic, as the case may be.

6. Analytic (Analysis) vs Synthetic (Synthesis) in Kant

To what extent can Kant indeed be considered an 'heir to the constructional sense' of the distinction between analysis and synthesis (*LLGI* 205)? According to Hintikka,

> (o)n two different occasions, Kant states as clearly as he can (which is not always *very* clearly) that his distinction between analysis and synthesis is modelled on the idea of a 'problematic' analysis, not on the directional sense of analysis and synthesis (*LLGI* 206).

The first passage alluded to here is a well known footnote from the *Prolegomena* in which Kant stresses that the distinction between analytic and synthetic judgements must not be confused with that between the analytic and synthetic methods:

> In consequence of the gradual advance of knowledge it is inevitable that certain expressions that have become classical and have been in use since the childhood of science, will be found inadequate and inappropriate, and that a certain new and more suitable usage will run into some danger of being confused with the old. Analytic method, in so far as it is opposed to the synthetic method is something quite different from an aggregate of analytic propositions (IV 276n).

Certainly the distinction between the analytic and synthetic methods is typified here by Kant according to the directional sense, and he goes on to add that it might be better to call this distinction that between the regressive and progressive methods (cf. IX 149). Hintikka's conclusion that in applying the distinction analytic–synthetic to judgements or propositions Kant in consequence appeals to the (old geometrical and, according to Hintikka, in Kant's time still current) constructional or problematic sense of this distinction is nevertheless a *non sequitur*. Consideration ought at least to be given in this case too to the possibility of taking Kant at his word; that is, it cannot be excluded beforehand that Kant's opposition between analytic and synthetic judgements really introduces a certain new usage of some old terms.

The second passage to which Hintikka alludes is also found in a footnote; this time it is at the beginning of the *Inaugural Dissertation* of the year 1770. There Kant distinguishes between quantitative and qualitative synthesis (and analysis):

> A double meaning is commonly assigned to the words 'analysis' and 'synthesis'. Thus synthesis is either *qualitative*, in which case it is a progression through a series of things which are subordinate to each other, the progression advancing from the ground to that which is grounded, or the synthesis is *quantitative*, in which case it is a progression within a series of things which are co-ordinate with each other, the progression advancing from a given part, through parts complementary to it, to the whole (II 388n; cf. XVI 16, XXIV 291).

Qualitative analysis and synthesis pertain to the relation of cause and effect and are placed by Hintikka self-evidently (and correctly) in the line of the directional interpretation of analysis and synthesis. Quantitative analysis and synthesis pertain to the relation of part-whole or simple-composite which Kant connects in this passage with the condition of time (II 387); time is also what the critical Kant affirms to be 'the medium of all synthetic judgements' (A 155/B 194).

However, when Kant goes on to conclude the footnote in question with the announcement: 'Here (i.e. in his *Inaugural Dissertation*) we use both 'synthesis' and 'analysis' only in their second (i.e. quantitative) sense', Hintikka again states as if it is self-evident: 'It is obvious that this statement assimilates Kant's notion of analysis once again to the notion of problematic (i.e., constructional) analysis discussed earlier' (*LLGI* 206). But why this would be obvious Hintikka nowhere makes plain.

To my mind this second passage should be viewed in a somewhat different light. For Kant here refers to both qualitative and quantitative synthesis as a progression (*progressus*); similarly, he speaks of the corresponding forms of analysis as a regression (*regressus*). The use of precisely these terms suggests strongly that not only the qualitative but also the quantitative interpretation of 'analysis' and 'synthesis' ought to be understood in terms of the analytic–synthetic distinction in its directional sense (cf. IV 276n!). Moreover, Kant takes up the distinction between quantitative and qualitative analysis and synthesis in a context in which he is not concerned at all with demonstration or proof. Consider also the central role of the notion of subordination (*subordinatio*), on the one hand, and the notion of coordination (*coordinatio*) on the other; both notions are connected by Kant with concepts in the first place.[33] From the perspective of an Aristotelian science this implies that here Kant does not connect the distinctions of analysis and synthesis directly or primarily with method as a method of proof, i.e. the demonstrative ordering of propositions according to the proof postulate. Analysis and synthesis, be it quantitative or qualitative, is used in the *Inaugural Dissertation* in its broad directional sense.

[33]Kant's distinction between a quantitative and a qualitative form of analysis and synthesis seems to go back to at least Boethius's very influential treatise *De Divisione*. Boethius distinguishes two forms of division. First there is the division of a whole into its parts, which he presents as a division in respect to quantity. In this sense a roof is part of a house, but Cato, Virgil and Cicero should be parts of *man* too; these individuals should be viewed as coordinate to each other. Secondly there is the division of a genus into its species; this division is carried out in respect to quality and is related to the notion of subordination: a species is subordinated to its genus. N. Kretzmann and E. Stump (eds), *The Cambridge Translation of Philosophical Texts*, Vol. I: *Logic and the Philosophy of Language* (Cambridge University Press, 1988), pp. 11–39, see p. 17. Distinctions in line with that of Boethius are frequently found in medieval and early modern philosophy. Cf. N. Kretzmann, 'Syncategoremata, exponibilia, sophismata', in N. Kretzmann, A. Kenny and J. Pinborg (eds), *The Cambridge History of Later Medieval Philosophy* (Cambridge University Press, 1982), pp. 211–245, see pp. 237–239. Note also that Zabarella, the leading logician of the school of Padua, distinguishes two kinds of the resolutive or analytic method; the first goes from a whole to its parts, the second from an effect to its cause, from a species to the genus that grounds it. Cf. J. Zabarella, *De methodis liber quatuor*, ed. C. Vasoli (Bologna: CLUEB, 1985), Book III, chap. XVI.

In summary, both the passages to which Hintikka appeals in no way state clearly that Kant's 'distinction between analysis and synthesis is modelled on the idea of a "problematic" analysis, not on the directional sense of analysis and synthesis' (*LLGI* 206; cf. *KK* 171, 182n13).

Hintikka interprets the Kantian distinction between the analytic and synthetic methods consistently in terms of different directions of inferences or differences in types of inference (the constructional sense); in other words, he sees this distinction concerning method from the outset in the context of the proof postulate of an Aristotelian science. That such an approach definitely leaves insufficient room for doing justice to Kant's use of analysis and synthesis can likewise be shown from the theses which were formulated in Section 3 to characterize the so-called 'Beth–Hintikka-reconstruction'.

Theses 2 and 3 allude to strong similarities between certain passages from the first *Kritik*, namely the part about the transcendental doctrine of method, and passages from the pre-critical treatise *Über die Deutlichkeit*. At issue is the contrast between the method of mathematics and that of philosophy or metaphysics; these differences are concentrated mainly on the status of definitions and on the method of proof. Kant takes up these subjects one after the other in the first sections of *Über die Deutlichkeit*:

> §1. Mathematics arrives at all its definitions synthetically, whereas philosophy arrives at all its definitions analytically (II 276; cf. XXIV 130–134).
>
> §2. Mathematics, in its analyses (*Auflösungen*), proofs and inferences examines the universal under signs *in concreto*; philosophy examines the universal by means of signs *in abstracto* (II 278).

I am not concerned at the moment to consider extensively this difference in method between mathematics and philosophy and to compare it with what Kant says about it in the first *Kritik*.[34] For the moment I am satisfied to have established *prima facie* that in *Über die Deutlichkeit* '*Analysis*', '*analytisch*', '*Synthesis*' and '*synthetisch*' are only spoken of in connection with definitions (or concepts); this approach—which is fully consistent with the broad directional interpretation of analysis and synthesis—appears again in the first *Kritik* where we read:

> ...the former (i.e., philosophical definitions) can be obtained only by analysis (...), the latter (i.e., mathematical definitions) are produced synthetically (A 730/B 758; cf. IX 141).

In contrast, the terms in question do not appear at all when Kant speaks about the differences between demonstrations in mathematics and philosophy. Neither *Über die Deutlichkeit* nor the pertinent passages from the first *Kritik* justify the conclusion that Kant directly connects the distinction between the

[34]See De Jong, *op. cit.*, note 21.

method of proof of mathematics and that of philosophy with the distinction between analysis and synthesis (A 734–735/B 762–763; II 278–279).[35]

One must conclude that Hintikka has not succeeded in supporting his thesis that Kant uses the distinction between analysis and synthesis in its constructional sense. The preceding has demonstrated the contrary to be the case, namely, that Kant is a heir of the (broad) directional interpretation of the distinction between analysis and synthesis.

7. Analytic and Synthetic Judgements in Kant

Now, all this suggests that the theses we presented in Section 4 as a rough characterization of Hintikka's interpretation of Kant should be reviewed critically. At issue is the validity and precise meaning of thesis 1. Hintikka's impressive historical and systematic interpretation of Kant's theory of mathematical method, at least insofar as it is presented in broad outline in theses 2, 3 and 4, is not in question here. The point of my argument concerns the way in which Hintikka self-evidently connects these theses regarding, in particular, the method of proof of mathematics with Kant's use of the paired terms 'analytic'–'synthetic' and 'analysis'– 'synthesis', or, as the case may be, interprets this method of proof with the help of these terms (recall his speaking of analytic vs synthetic argument steps). I have argued thus far that Kant introduces such a linkage neither in his critical nor in his pre-critical period. With that, thesis 1 as Hintikka understands it is left without any basis.

What remains then is the question concerning the connection between the analytic and synthetic methods in the directional sense on the one hand and, on the other, the distinction between the analytic and synthetic judgements in the critical Kant—not so much in the historical as in a systematic respect. Tied to this, too, is the question of the precise place and role of the PNC as 'the universal and completely sufficient principle of all analytic knowledge' (A 151/B 191). And that leads me back to the problematic passage B 14 that was cited in Section 1.

Elsewhere I have argued that a correct apprehension of Kant's view of the opposition between analytic and synthetic judgements requires that a careful distinction be made between his notion of analyticity and his (epistemological) criterion for analyticity.[36] Restricted to (affirmative) subject-predicate judgements, Kant's notion of analyticity can be characterized as follows:

[35]Notice also in this regard that, otherwise than in the first *Kritik*, Kant in the pre-critical writing *Über die Deutlichkeit* (still) does not bring the synthetic definitions of mathematics into connection at all with (*a priori*) constructions or intuitions; he speaks of synthesis only as an 'arbitrary combination of concepts' (II 276, 280).

[36]Cf. W. R. de Jong, 'Kant's Analytic Judgements and the Traditional Theory of Concepts', *Journal of the History of Philosophy* **33** (1995), 613–641.

> The judgement *S is P* (or *All S are P*, or even *Some S are P*) is analytic means that the predicate concept *P* of this judgement is contained in (*ist enthalten in*) its subject-concept *S* (cf. A 6–7/B 10; IV 266, XVI 629).

And consequently it is so that (true) (subject-predicate) judgements which are not analytic are synthetic judgements.

Moreover, analytic judgements are according to Kant *eo ipso a priori* or necessarily true:

> ..., if the judgement is analytic,...., its truth can always be adequately known in accordance to (*erkannt werden nach*) the principle of contradiction (A 150/B 190; cf. IV 267).

Furthermore, this condition is not only a necessary but also a sufficient condition of analyticity:

> (synthetic judgements *a posteriori* and *a priori*) agree in that they can never originate according to (*entspringen nach*) the principle of analysis alone, namely the principle of contradiction (IV 267; cf. A 151–152/B 191).

In other words, the PNC—which, as *the* fundamental principle of Kant's general logic, corresponds to provability within this logic—determines an epistemological criterion for analyticity:

> A (subject-predicate) judgement is analytic (analytically true) if and only if it can be demonstrated according to the principle of contradiction alone.[37]

One must accordingly conclude that modern interpretations of Kant's notion of analyticity in the line of Frege, but also of Hintikka, are oriented primarily to Kant's criterion of analyticity rather than to his characterization of analyticity that precedes it.

Yet what is the exact relation between this criterion and Kant's notion of analyticity? It is precisely here that the notion of analysis comes to the fore:

> All analytic judgements rest wholly on the principle of contradiction, and it is in their nature to be knowledge *a priori*,... For because the predicate of an affirmative analytic judgement has already been thought in (i.e., is contained in) the concept of the subject, it cannot be denied of the subject without contradiction.
>
> ..., e.g. gold is a yellow metal; for to know this I need no further experience outside my concept of gold, which contained that this body is yellow and metal; for this is what constituted my concept, and I needed to do nothing except analyse it.... (IV 267)

To see clearly that a predicate-concept is contained in a subject-concept one has to analyse the latter concept. 'Analytic judgements need an analysis and in this way they serve to explain concepts' (XX 322). It is in the line of this analysis of concepts that Kant calls the PNC 'the principle of analysis'.

[37]The meaning of the phrase 'according to' in this context can be reconstructed in a more precise way. Cf. *op. cit.*, note 36. Note however that Kant himself does not have a clear *terminus technicus* to characterize the way in which the principle of contradiction functions in demonstrations or proofs; phrases such as 'according to' and 'in accordance with' correspond with divergent, often not very precise, formulations of Kant himself.

With the application of the paired terms 'analytic'–'synthetic' to judgements or propositions *tout court* the critical Kant introduces a new usage of these expressions. On this point we may take Kant at his word (cf. IV 276n). This new use is motivated primarily by the traditional directional sense of analysis as applied to concepts. An analytic judgement in Kant is a judgement in which the predicate concept is contained in the subject concept; the presence of this relation of containment can be made explicit by an analysis of concepts. Synthetic judgements are simply non-analytic judgements, i.e. judgements in which the predicate concept is not contained in the subject concept as can be made explicit by a (complete) analysis of the concepts concerned. This also means that the distinction between analytic and synthetic judgements cannot be explained in terms of a contrast between analytic and synthetic methods of proof (in the directional sense).[38] Note that the fundamental propositions of mathematics are synthetic (*a priori*) according to Kant.

I return to 'the very perplexing passage' cited at the beginning of this paper (B 14; IV 268). The above will have made clear that Hintikka's interpretation of this passage is somewhat less than acceptable. But how in that case can this passage—in which Kant opposes Leibniz, among others—be understood in a way consistent with what Kant says elsewhere about what is proper to the method of proof in mathematics? The problems are concentrated around two phrases:

> ... as it was found that all mathematical inferences proceed in accordance with (*fortgehen nach*) the principle of contradiction (which the nature of all apodeictic certainty requires),...

and,

> ... though a synthetic proposition can indeed be discerned in accordance with (*eingesehen nach*) the principle of contradiction, this can only be if another synthetic proposition is presupposed, and if it can then be apprehended as following from the other proposition;...

To begin with, I do not believe that Kant really weighed his words very carefully here. The possibility has to be left open that this passage is somewhat vague or equivocal (see also note 37). For all that, I believe it is possible to arrive at a fairly satisfying interpretation after all.

According to Leibniz the PNC is sufficient to demonstrate all mathematical propositions. And Kant too acknowledges that this (regulative) principle, being the core of 'general logic', has to be accepted in any science, and thus in arithmetic or geometry too. However, according to Kant this principle constitutes a criterion of analyticity. Claiming against Leibniz (and Hume) that real mathematical propositions are synthetic, Kant has to state:

[38]This does not detract from the fact that Kant's 'certain new and more suitable usage (of the expressions 'analytic' and 'synthetic') will run into some danger of being confused with the old (i.e. these expressions in their (broad) directional sense)' (cf. IV 276n).

> *if* a synthetical proposition is demonstrated using the PNC, it has to be demonstrated as following by the PNC from an other synthetic proposition.

I think this is not only a fairly kind, but in essence also a correct interpretation of what Kant has in mind in the second phrase cited above. In the final instance there is an implication, for Euclidean geometry, for example, that in any case its axioms must be deemed synthetic. Yet that is not the end of the matter.

In the beginning of the part on transcendental analysis in the first *Kritik*, Kant discusses the role of the PNC as a criterion of truth. In doing so he distinguishes emphatically between the positive and the negative employment of this principle. The former involves its use 'for the knowing of truth'; this is the use of the PNC as a criterion of analyticity, which has already been discussed above. Concerning the latter Kant states:

> The proposition that no predicate contradictory of a thing can belong to it, is entitled the principle of contradiction, and is a universal, though merely negative, criterion of all truth. For this reason it belongs only to logic. It holds of knowledge, merely as knowledge in general, irrespective of content, and asserts that the contradiction completely cancels and invalidates it (A 151/B 190).

And Kant proceeds directly to make clear that in connection with the synthetic portion of knowledge, the PNC can only be used negatively:

> and in regard to the truth of this kind of knowledge we can never look to the above principle (i.e. the PNC) for any positive information, though, of course, since it is inviolable, we must always be careful to conform to it (A 152/B 191).

In B 14 Kant argues against Leibniz, among others, that mathematical knowledge is synthetic, not analytic. In the light of the above this means he holds that in mathematics the PNC can only be invoked as a negative 'criterion of truth'. The view he energetically opposes is manifestly that of the PNC as a positive criterion of truth for mathematical knowledge. Kant in principle already lays this notion to rest by asserting that the PNC only offers the possibility of inferring synthetic propositions from synthetic propositions.

This brings us to the first phrase, the real *pièce de résistance* in the passage in question. To begin with, a certain view of others, including Leibniz and Wolff, is described here. Beyond that Kant himself seems to accept here that the nature of all apodeictic certainty requires 'that all mathematical inferences proceed in accordance with (*fortgehen nach*) the principle of contradiction'. But he can hardly have had in mind here the PNC as a positive criterion of truth. Kant's entire argument would collapse in that case. An interpretation of his own position consistent with the PNC as a negative criterion of truth would, however, serve a number of ends.

First, the argument of B 14 is convincing and, for the rest, susceptible of harmonization with what Kant says elsewhere about the role of the PNC.

Secondly, and at least as important, when interpreted in this way the phrase in question is compatible with Kant's description of mathematical demonstration as a reasoning *in concreto*. As we saw, in an Aristotelian science one usually distinguishes between common (general) and proper (specific) principles. Kant introduces in his philosophy of mathematics such a distinction for the (regulative) principles constituting the proof postulate. In geometry (or mathematics) there would be besides the (regulative) principles constituting 'general logic' (with the PNC at its centre), also still one or more specific principles. Mathematical inference labelled by Kant as a reasoning '*in concreto*' is of this sort. In the 'Beth–Hintikka-reconstruction' these domain-specific rules—which Hintikka refers to as 'synthetic argument steps'—are related to, or even identified with, the rule of existential instantiation and/or the rule of universal generalization. These rules are part of modern predicate logic which is a consistent theory, i.e. does not imply a contradiction.

Given this reading of B 14 (see IV 268) in terms of the PNC as a 'negative criterion of truth', the phrase in question is relieved of its 'perplexing' character. One can of course still criticize Kant for not having distinguished adequately here between the positive and the negative employment of the PNC.

8. Conclusion

In this paper Hintikka's well-known interpretation of Kant's philosophy of mathematics was subjected to critical scrutiny. It turned out that Kant's distinction between analytic and synthetic judgements (*a priori*) is not based on the contrast between analysis and synthesis in its constructional sense, as Hintikka claims. More positively, it is argued here that this distinction is closely connected by Kant to a specific concept of *a priori* science—namely, the Aristotelian model of science. In addition, in discussing method and/or definitions (both the critical and the pre-critical) Kant used the distinction between analysis and synthesis in a (broad) directional sense. What Hintikka calls 'synthetic argument steps' correspond to specific rules of inference outside Kant's general logic. These rules only hold within the domain of a priori intuitable, mathematical objects. With the distinction between analytic and synthetic judgements, the critical Kant did indeed introduce—as he himself claims—a new way of using the paired words 'analytic'–'synthetic', but one that still lies in the line of the use of these terms in their directional sense. A careful comparison of Kant's conceptions from the first *Kritik* and the *Prolegomena*, on the one hand, with the ideas of the pre-critical Kant as expressed in *Über die Deutlichkeit*, on the other hand, produces, as I see it, compelling arguments for the interpretation defended here of Kant's distinction between analytic and synthetic judgements within his philosophy of mathematics.